1 Prandtl layer parameterisation in PALM

The friction velocity u_* is a velocity scale that is defined by the relation

$$u_* = (|\tau/\overline{\rho}|)^{\frac{1}{2}}, \tag{1.1}$$

where τ is the Reynolds stress and ρ is the air density. Using the surface kinematic momentum fluxes in the x and y directions $\left(-\overline{u'w'}, -\overline{v'w'}\right)$ to represent surface stress, the friction velocity can be written as

$$u_* = \left((-\overline{u'w'})^2 + (-\overline{v'w'})^2\right)^{\frac{1}{4}}. \tag{1.2}$$

Based on the definition of the friction velocity u_* the vertical turbulent momentum flux $\overline{u'w'}$ can be determined from

$$u_*^2 \cos(\alpha_0) = -\overline{u'w'}, \tag{1.3}$$

while the vertical turbulent momentum flux $\overline{v'w'}$ can be determined from

$$u_*^2 \sin(\alpha_0) = -\overline{v'w'}. \tag{1.4}$$

The angle α_0, that is assumed to be constant in the Prandtl layer, is the angle between the x-direction and the direction of the mean horizontal wind and can be evaluated by

$$\alpha_0 = \arctan\left(\frac{\overline{v}(z_p)}{\overline{u}(z_p)}\right). \tag{1.5}$$

According to the similarity theory of Monin and Obukhov the following relationship for the profile of the mean horizontal wind is valid

$$\begin{aligned}
\frac{\partial |\overline{\vec{v}}|}{\partial z} &= \frac{u_*}{\kappa z}\phi_m\left(\frac{z}{L}\right) \\
&= u_*\frac{1}{\kappa z}\phi_m\left(\frac{z}{L}\right).
\end{aligned} \tag{1.6}$$

In equation 1.6 L denotes the Monin Obukhov length and κ denotes the von Karman constant, while ϕ_m denotes the profile or Dyer-Businger functions for momentum:

$$\phi_m = \begin{cases} 1 + 5\text{Rif} & \text{if Rif} > 0, \\ 1 & \text{if Rif} = 0, \\ (1 - 16\text{Rif})^{-\frac{1}{4}} & \text{if Rif} < 0. \end{cases} \tag{1.7}$$

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Rif denotes the dimensionless Richardson flux number.

By integrating equation 1.6 over z from z_0 to a height z the following relationship for the friction velocity can be deduced:

$$
u_* = \begin{cases} \dfrac{|\vec{\overline{v}}|\kappa}{\left(\ln\left(\frac{z}{z_0}\right)+5\mathrm{Rif}\left(\frac{z-z_0}{z}\right)\right)} & \text{if Rif} \geq 0, \\[4mm] \dfrac{|\vec{\overline{v}}|\kappa}{\ln\left(\frac{z}{z_0}\right)-\ln\left(\frac{(1+A)^2(1+A^2)}{(1+B)^2(1+B^2)}\right)+2(\arctan(A)-\arctan(B))} & \text{if Rif} < 0 \end{cases}
\tag{1.8}
$$

with

$$
A = (1 - 16\mathrm{Rif})^{\frac{1}{4}}
\tag{1.9}
$$

and

$$
B = \left(1 - 16\mathrm{Rif}\frac{z_0}{z}\right)^{\frac{1}{4}}.
\tag{1.10}
$$

In fact, equation 1.8 is used in PALM to determine the friction velocity u_*.

The following paragraph is a short digression dealing with the integration of equation 1.6 that finally leads to equation 1.8. The integration is only shown for the profile function for unstable stratification. According to PAULSON (1970) the general result of an integration of $\phi = \frac{\kappa z'}{u_*}\frac{\partial \overline{u}}{\partial z'}$ over z' from z_0 to z is

$$
\overline{u}(z) = \frac{u_*}{\kappa}\left[\ln\left(\frac{z}{z_0}\right) - \Psi\right]
\tag{1.11}
$$

with

$$
\Psi = \int_{\frac{z_0}{L}}^{\frac{z}{L}} d\left(\frac{z'}{L}\right)\frac{1 - \phi\left(\frac{z'}{L}\right)}{\frac{z'}{L}}.
\tag{1.12}
$$

Applying the profile function for unstable stratification, $\phi = \left(1 - 16\frac{z}{L}\right)^{-\frac{1}{4}}$, Ψ can be deter-

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mined as follows:

$$
\begin{aligned}
\Psi &= \int_{\frac{z_0}{L}}^{\frac{z}{L}} d\left(\frac{z'}{L}\right) \frac{1 - \left(1 - 16\frac{z'}{L}\right)^{-\frac{1}{4}}}{\frac{z'}{L}} \quad\text{---}\quad \text{substitution: } x = \frac{1}{\phi\left(\frac{z'}{L}\right)} \\
&= \int_{\left(1-16\frac{z_0}{L}\right)^{\frac{1}{4}}}^{\left(1-16\frac{z}{L}\right)^{\frac{1}{4}}} dx \left(4\frac{x^2 - x^3}{1 - x^4}\right) \\
&= \int_{\left(1-16\frac{z_0}{L}\right)^{\frac{1}{4}}}^{\left(1-16\frac{z}{L}\right)^{\frac{1}{4}}} dx \left(2\left(\frac{1}{1+x} + \frac{x}{1+x^2} - \frac{1}{1+x^2}\right)\right) \\
&= \left[2\ln\left(\frac{1+x}{2}\right) + \ln\left(\frac{1+x^2}{2}\right) - 2\arctan(x)\right]_{\left(1-16\frac{z_0}{L}\right)^{\frac{1}{4}}=B}^{\left(1-16\frac{z}{L}\right)^{\frac{1}{4}}=A} \\
&= \ln\left(\frac{(1+A)^2}{(1+B)^2}\frac{1+A^2}{1+B^2}\right) - 2\left(\arctan(A) - \arctan(B)\right)
\end{aligned}
\tag{1.13}
$$

Making use of equation 1.5 and 1.6, it is also possible to deduce a relationship for the profile of the mean u-component of the wind velocity

$$
\begin{aligned}
\frac{\partial \overline{u}}{\partial z} &= \frac{\partial \left|\overline{\vec{v}}\right|}{\partial z} \cos\left(\alpha_0\right) \\
&= \frac{u_*}{\kappa z}\phi_m\left(\frac{z}{L}\right)\cos\left(\alpha_0\right) \\
&= \frac{1}{u_*}u_*^2\cos\left(\alpha_0\right)\frac{1}{\kappa z}\phi_m\left(\frac{z}{L}\right) \\
&= \frac{-\overline{u'w'}}{u_*}\frac{1}{\kappa z}\phi_m\left(\frac{z}{L}\right)
\end{aligned}
\tag{1.14}
$$

and accordingly a relationship for the profile of the mean v-component of the wind velocity

$$
\begin{aligned}
\frac{\partial \overline{v}}{\partial z} &= \frac{\partial \left|\overline{\vec{v}}\right|}{\partial z} \sin\left(\alpha_0\right) \\
&= \frac{u_*}{\kappa z}\phi_m\left(\frac{z}{L}\right)\sin\left(\alpha_0\right) \\
&= \frac{1}{u_*}u_*^2\sin\left(\alpha_0\right)\frac{1}{\kappa z}\phi_m\left(\frac{z}{L}\right) \\
&= \frac{-\overline{v'w'}}{u_*}\frac{1}{\kappa z}\phi_m\left(\frac{z}{L}\right).
\end{aligned}
\tag{1.15}
$$

As the right-hand sides of equation 1.14 and 1.15 differ only by prefactors that are (assumed to be) constant within the Prandtl layer from the right-hand side of equation 1.6, we can

directly make use of the integration that led to equation 1.8 in order to obtain

$$C = \frac{-\overline{u'w'}}{u_*} = \begin{cases} \dfrac{\overline{u}\kappa}{\left(\ln\left(\frac{z}{z_0}\right)+5\mathrm{Rif}\left(\frac{z-z_0}{z}\right)\right)} & \text{if Rif} \geq 0, \\ \dfrac{\overline{u}\kappa}{\ln\left(\frac{z}{z_0}\right)-\ln\left(\frac{(1+A)^2(1+A^2)}{(1+B)^2(1+B^2)}\right)+2(\arctan(A)-\arctan(B))} & \text{if Rif} < 0. \end{cases} \quad (1.16)$$

and

$$D = \frac{-\overline{v'w'}}{u_*} = \begin{cases} \dfrac{\overline{v}\kappa}{\left(\ln\left(\frac{z}{z_0}\right)+5\mathrm{Rif}\left(\frac{z-z_0}{z}\right)\right)} & \text{if Rif} \geq 0, \\ \dfrac{\overline{v}\kappa}{\ln\left(\frac{z}{z_0}\right)-\ln\left(\frac{(1+A)^2(1+A^2)}{(1+B)^2(1+B^2)}\right)+2(\arctan(A)-\arctan(B))} & \text{if Rif} < 0. \end{cases} \quad (1.17)$$

Both equations, 1.16 and 1.17, are used in PALM. In order to get an information on the turbulent vertical momentum fluxes within the Prandtl layer, equation 1.16 and equation 1.17 need only to be multiplied by -1 and the friction velocity u_*, so that finally the followings two steps have to be executed in PALM:

$$\overline{u'w'} = -Cu_* \quad (1.18)$$

and

$$\overline{v'w'} = -Du_*. \quad (1.19)$$

In case that no near-surface heat flux has been prescribed by the user of PALM, the near-surface heat flux $\overline{w'\Theta'}_0$ is evaluated from

$$\overline{w'\Theta'} = -u_*\theta_*. \quad (1.20)$$

Here, θ_* is the so-called characteristic temperature for the Prandtl layer. In case of no preset near-surface heat flux the characteristic temperature θ_* is determined in PALM from the integrated version of the profile function for the potential temperature. According to the similarity theory of Monin and Obukhov the following relationship for the vertical gradient of the potential temperature is valid:

$$\frac{\partial\overline{\Theta}}{\partial z} = \frac{\theta_*}{\kappa z}\phi_h. \quad (1.21)$$

In equation 1.21 ϕ_h denotes the profile or Dyer-Businger functions for temperature:

$$\phi_h = \begin{cases} 1 + 5\mathrm{Rif} & \text{if Rif} > 0, \\ 1 & \text{if Rif} = 0, \\ (1 - 16\mathrm{Rif})^{-\frac{1}{2}} & \text{if Rif} < 0. \end{cases} \quad (1.22)$$

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By integrating equation 1.21 over z from z_0 to a height z the following relationship for the characteristic temperature in the Prandtl layer can be deduced:

$$\theta_* = \begin{cases} \dfrac{\kappa\left(\overline{\Theta}(z)-\overline{\Theta}(z_0)\right)}{\ln\left(\frac{z}{z_0}\right)+5\mathrm{Rif}\left(\frac{z-z_0}{z}\right)} & \text{if Rif} \geq 0, \\[4mm] \dfrac{\kappa\left(\overline{\Theta}(z)-\overline{\Theta}(z_0)\right)}{\ln\left(\frac{z}{z_0}-2\ln\left(\frac{1+A}{1+B}\right)\right)} & \text{if Rif} < 0 \end{cases} \tag{1.23}$$

with

$$A = \sqrt{1 - 16\mathrm{Rif}} \tag{1.24}$$

and

$$B = \sqrt{1 - 16\mathrm{Rif}\frac{z_0}{z}}. \tag{1.25}$$

Note, that the temperature at the height of the roughness length that is required for the evaluation of θ_* in equation 1.23 is saved in PALM at the first vertical grid level of the temperature with index k=0. The height that is assigned to this vertical grid level is $-0.5\Delta z$, where Δz is the grid length in the vertical direction. In case of a near-surface heat flux $\overline{w'\Theta'}_0$ that has been prescribed by the user, the evaluation of θ_* is not needed for the integration of the model. Instead θ_* is only determined for the evaluation of statistics of the turbulent flow. In that case it is simply derived by a transformation of equation 1.20 into

$$\theta_* = -\frac{\overline{w'\Theta'}_0}{u_*}. \tag{1.26}$$

In case that PALM is run in its moist version, the evaluation of the characteristic humidity for the Prandtl layer is evaluated in a way correspondent to that of the determination of the characteristic temperature.