Basic Equations

PALM group

Institute of Meteorology and Climatology, Leibniz Universität Hannover

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Basic equations, Unfiltered

Basic equations, Unfiltered

- Navier-Stokes equations

$$\rho \frac{\partial u_i}{\partial t} + \rho u_k \frac{\partial u_i}{\partial x_k} = -\frac{\partial p}{\partial x_i} - \rho \varepsilon_{ijk} f_j u_k - \rho \frac{\partial \phi}{\partial x_i} + \mu \left\{ \frac{\partial^2 u_i}{\partial x_k^2} + \frac{1}{3} \frac{\partial}{\partial x_i} \left(\frac{\partial u_k}{\partial x_k} \right) \right\}$$

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- First principle

$$\rho\frac{\partial T}{\partial t}+\rho u_k\frac{\partial T}{\partial x_k}=\mu_{\mathrm{h}}\frac{\partial^2 T}{\partial x_k^2}+Q$$

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- Continuity equation

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho u_k}{\partial x_k}$$

Boussinesq Approximation

Boussinesq Approximation

- Splitting thermodynamic variables into a basic state ψ_0 and a variation ψ^*

$$
\begin{aligned}
T(x,y,z,t) &= T_0(x,y,z) && +T^*(x,y,z,t) \\
p(x,y,z,t) &= p_0(x,y,z) && +p^*(x,y,z,t) \\
\rho(x,y,z,t) &= \rho_0(z) && +\rho^*(x,y,z,t); && \psi^* << \psi_0
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- Hydrostatic equilibrium, geostrophic wind (not included in Boussinesq)

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\frac{\partial p_0}{\partial z} = -g\rho_0 \qquad \frac{1}{\rho_0}\frac{\partial p_0}{\partial x} = -f v_{\mathrm{g}}, \qquad \frac{1}{\rho_0}\frac{\partial p_0}{\partial y} = f u_{\mathrm{g}}
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- Equation of state

$$p = \rho R T \rightarrow \ln p = \ln \rho + \ln R + \ln T \rightarrow \frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

$$\frac{\Delta p}{p_0} \approx \frac{\Delta \rho}{\rho_0} + \frac{\Delta T}{T_0} \rightarrow \frac{p^*}{p_0} \approx \frac{\rho^*}{\rho_0} + \frac{T^*}{T_0} \qquad \frac{\rho^*}{\rho_0} \approx -\frac{T^*}{T_0}$$

Continuity Equation

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$$\frac{\partial \rho_0(z)}{\partial t} = -\frac{\partial \rho_0(z) u_k}{\partial x_k} \qquad \frac{\partial \rho_0 u_k}{\partial x_k} = 0$$

anelastic approximation

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anelastic approximation

$$\rho_0 = const. \qquad \frac{\partial u_k}{\partial x_k} = 0$$

incompressible flow

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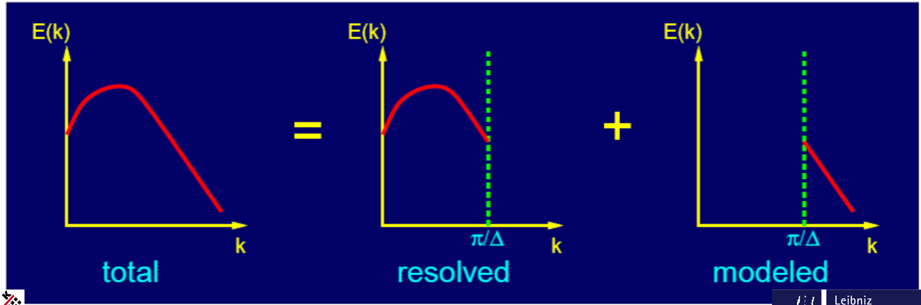
This set of equations is valid for almost all kind of CFD models!

LES - Scale Separation by Spatial Filtering (I)

- LES technique is based on scale separation, in order to reduce the number of degrees of freedom of the solution. $\boxed{\Psi(x_i,t) = \overline{\Psi}(x_i,t) + \Psi'(x_i,t)}$



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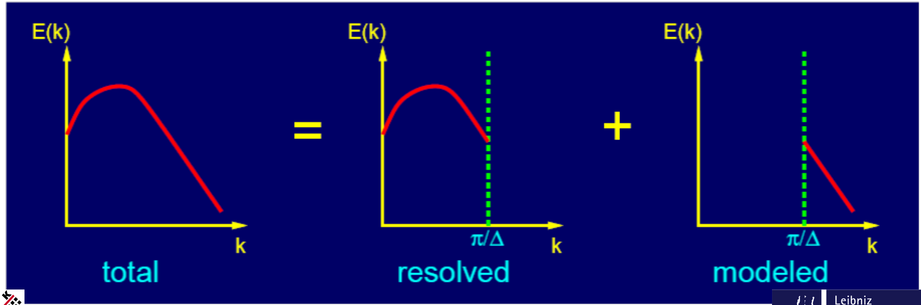

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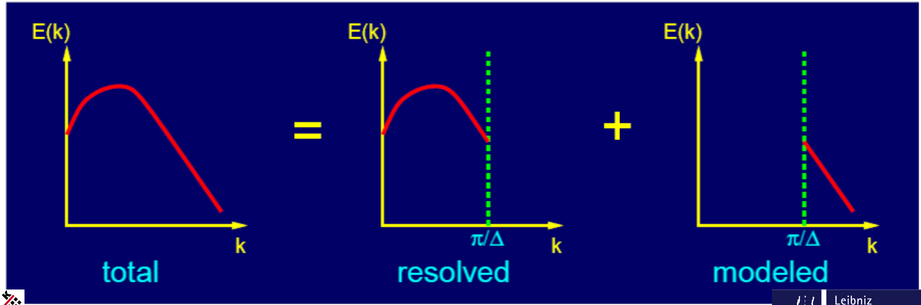

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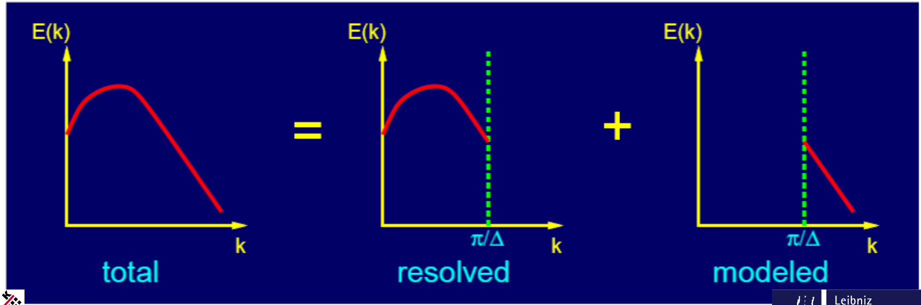
- These two categories of scales are seperated by defining a cutoff length Δ.



LES - Scale Separation by Spatial Filtering (II)

- The Filter applied is a spatial filter:

$$\overline{\Psi}(x_i) = \int_D G(x_i - x_i')\Psi(x_i')dx_i'$$

$$\overline{\Psi}'(x_i) = 0 \qquad but \qquad \overline{\overline{\Psi}} \neq \overline{\Psi}(x_i)$$

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- Filter applied to the nonlinear advection term:

$$\overline{u_k u_i} = \overline{(\overline{u_k} + u_k')(\overline{u_i} + u_i')} = \overline{\overline{u_k}\,\overline{u_i}} + \underbrace{\overline{\overline{u_k} u_i'} + \overline{u_k' \overline{u_i}}}_{C_{ki}} + \underbrace{\overline{u_k' u_i'}}_{R_{ki}}$$

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- Leonard proposes a further decomposition:

$$\overline{\overline{u_k}\,\overline{u_i}} = \overline{u_k}\,\overline{u_i} + \underbrace{\left(\overline{\overline{u_k}\,\overline{u_i}} - \overline{u_k}\,\overline{u_i}\right)}_{L_{ki}}$$

$$\overline{u_k u_i} = \overline{u_k}\,\overline{u_i} + L_{ki} + C_{ki} + R_{ki} = \overline{u_k}\,\overline{u_i} + \tau_{ki}$$

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- Volume-balance approach (Schumann, 1975) advantage: numerical discretization acts as a Reynolds operator

$$\Psi(V,t) = \frac{1}{\Delta x \cdot \Delta y \cdot \Delta z} = \int\int\int_V \Psi(V',t)dV'$$

$$\overline{\Psi'}(x_i) = 0 \quad \text{and} \quad \overline{\overline{\Psi}} = \overline{\Psi}$$

$$V = \left[x - \frac{\Delta x}{2}, x + \frac{\Delta x}{2}\right] \times \left[y - \frac{\Delta y}{2}, y + \frac{\Delta y}{2}\right] \times \left[z - \frac{\Delta z}{2}, z + \frac{\Delta z}{2}\right]$$

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Leibniz Universität Hannover

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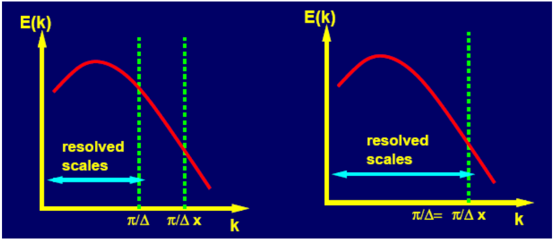
- Two possibilities:
  1. Pre-filtering technique ($\Delta x < \Delta$, explicit filtering)
  2. Linking the analytical filter to the computational grid ($\Delta x = \Delta$, implicit filtering)



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Literature:
Sagaut, P., 2001: Large eddy simulation for incompressible flows: An introduction. Springer Verlag, Berlin/Heidelberg/New York, 319 pp.
Schumann, U., 1975: Subgrid scale model for finite difference simulations of turbulent flows in plane channels and annuli. J. Comp. Phys., **18**, 376-404.

The Final Set of Equations (PALM)

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- Navier-Stokes equations:

$$\frac{\partial \overline{u_i}}{\partial t} = -\frac{\partial \overline{u_k}\,\overline{u_i}}{\partial x_k} - \frac{1}{\rho_0}\frac{\partial \overline{\pi}^*}{\partial x_i} - \varepsilon_{ijk} f_j \overline{u_k} + \varepsilon_{i3k} f_3 \overline{u}_{k_g} + g\frac{\overline{\theta} - \theta_0}{\theta_0}\delta_{i3} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_k^2} - \frac{\partial \tau^r_{ki}}{\partial x_k}$$

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subgrid-scale stresses (fluxes) to be parameterized in the SGS model:

$$\tau_{ki} = \overline{u_k u_i} - \overline{u_k}\,\overline{u_i}$$
$$H_k = \overline{u_k \theta} - \overline{u_k}\,\overline{\theta}$$
$$W_k = \overline{u_k q} - \overline{u_k}\,\overline{q}$$